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Local state probabilities for an infinite sequence of solvable lattice models

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Abstract. We present a new infinite sequence of solvable lattice models. They contrast strikingly with the eight-vertex solid-on-solid models and admit extra degrees of freedom for the local fluctuation variables. The exact one-point functions are obtained. The result is neatly described in terms of theta-function identities. Using their modular invariance, critical behaviour is studied and exponents evaluated.

1. Introduction

The study of exactly solvable models in two dimenions has entered a new phase through the exact evaluation of the one-point functions and the appearance of the conformal field theory (CFT). The restricted eight-vertex solid-on-solid (8VSOS) models of Andrews *et al* [1] realise the anomalous dimensions of minimal CFT [2] through the critical exponents obtained from the one-point functions [3]. The results have been extended to more general SOS models in [4, 5], yielding the anomalous dimensions of CFT having supersymmetries, Z_N invariance, etc. The analysis evolved in these works naturally leads to a correspondence principle between one-point functions of solvable lattice models and irreducible decomposition of characters for affine Lie algebras. Moreover, the corresponding principle explains the phenomenologically observed coincidence of the exponents for the 8VSOS models and minimal CFT.

The purpose of this paper is to add further exact results on the one-point functions, which we call the local state probabilities (LSP). The LSP $P(\lambda)$ by definition gives the probability that a local state λ_i on the lattice site i takes a given state λ. We introduce a series of new interaction-round-face models that contains those solved in [6] and study their critical behaviour through the exact evaluation of the LSP. Much the same as the 8VSOS models, our models are labelled by an integer L (denoted by r in [1]). However, they manifest significant differences from each other in various aspects.

The layout of the paper is as follows. In the next section we define our models and explain that the odd-L cases are identified with the models in [6]. The equivalence provides us with a solution of the star-triangle equations for general L. In § 3 we outline the calculation of the LSP for a regime of the parameters and give the results. This section is based on Baxter's corner transfer matrix method [7] and the mathematical techniques developed in [5]. In § 4, critical behaviour of the LSP is investigated and the exponents obtained. The last section is devoted to a summary and discussion.

2. The models

Consider a planar square lattice with a fluctuation variable λ_i associated with each site i. We shall call λ_i a state. Let L be an integer satisfying $L \geq 3$. We assume that each λ_i takes the following $(L+3)$ states:

$$\lambda_i \in \{0, \bar{0}, 1, 2, \ldots, L-1, L, \bar{L}\}. \tag{2.1}$$

We impose the conditions on the states that occupy the neighbouring lattice sites. These are best described by figure 1. In this diagram each vertex (open circle) stands for a local state λ_i. Two states are allowed to occupy adjacent lattice sites if the corresponding vertices are connected on the diagram. We shall call such a pair of states admissible. Let λ_i, λ_j, λ_k and λ_l be the states such that four pairs (λ_i, λ_j), (λ_j, λ_k), (λ_k, λ_l) and (λ_l, λ_i) are admissible. We assign a Boltzmann weight $w(\lambda_i, \lambda_j, \lambda_k, \lambda_l)$ for the state configuration $(\lambda_i, \lambda_j, \lambda_k, \lambda_l)$ round a face, where the sites i, j, k and l of the face are ordered anticlockwise from the south-west corner. We assume the following properties for the Boltzmann weights.

(i) 'Top-bottom' symmetry:

$$w(\lambda_i, \lambda_j, \lambda_k, \lambda_l) = w(\lambda_i^*, \lambda_j^*, \lambda_k^*, \lambda_l^*) \tag{2.2a}$$

where λ^* is defined by

$$\begin{aligned} &\lambda^* = L - \lambda \qquad && \text{if } \lambda \neq \bar{0}, \bar{L} \\ &(\bar{0})^* = \bar{L} \qquad && (\bar{L})^* = \bar{0}. \end{aligned} \tag{2.2b}$$

(ii) 'Replica' symmetry:

$$w(\lambda_i, \lambda_j, \lambda_k, \lambda_l) = w(\bar{\lambda}_i, \bar{\lambda}_j, \bar{\lambda}_k, \bar{\lambda}_l) \tag{2.3a}$$

where $\bar{\lambda}$ is defined by

$$\begin{aligned} &\bar{\lambda} = \lambda \qquad && \text{if } \lambda \neq 0, \bar{0}, L, \bar{L} \\ &\overline{(0)} = \bar{0} \qquad && \overline{(L)} = \bar{L} \qquad \overline{(\bar{0})} = 0 \qquad \overline{(\bar{L})} = L. \end{aligned} \tag{2.3b}$$

(iii) Diagonal exchange symmetry:

$$\begin{aligned} w(\lambda_i, \lambda_j, \lambda_k, \lambda_l) &= w(\lambda_i, \lambda_l, \lambda_k, \lambda_j) \\ &= w(\lambda_k, \lambda_j, \lambda_i, \lambda_l). \end{aligned} \tag{2.4}$$

Our model is an interaction-round-face model [7] specified by these conditions on the state variables and the Boltzmann weights. We note that the $L = 3$ case is equivalent to odd-height sectors of the SOS models in [4, 5] with $(L, N) = (6, 2)$.

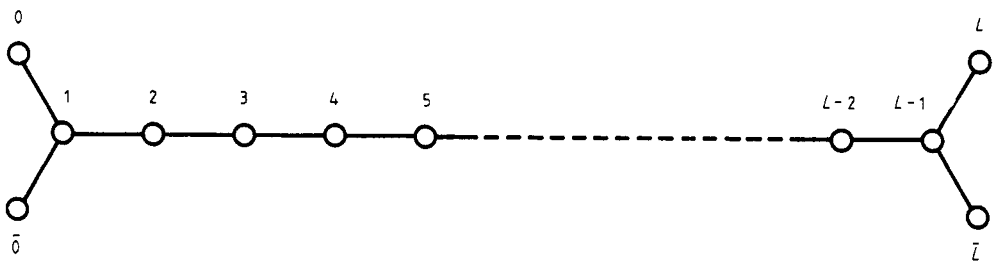

Figure 1. Diagram for our model. Each vertex (open circle) corresponds to a local state. Two states can occupy nearest-neighbour lattice sites if they are connected on the diagram.

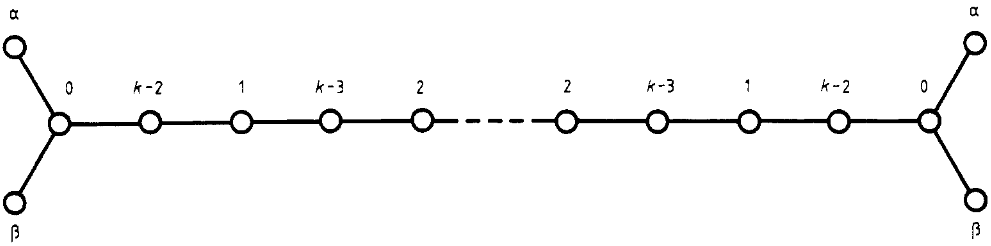

Figure 2. Diagram for the model in [6]. Each vertex (open circle) corresponds to a local state (called 'spin' in [6]). Two states can occupy nearest-neighbour lattice sites if they are connected on the diagram.

The solution of the star-triangle equations is obtained by using relations with the model solved in [6] which we shall now explain. Let us first assume that L is odd and set $L=2k-1$ $(k\geqslant 2)$. We introduce a 'spin' variable σ_i that takes $(k+1)$-spin states: $\sigma_i \in \{0, 1, 2, \ldots, k-2, \alpha, \beta\}$. By figure 2 we define a map from the state variable λ_i to the spin variable σ_i. It shows the images of the λ_i in the corresponding positions on figure 1.

This transforms the odd-L model to the $(k+1)$-spin state model in [6] and hence gives us the solution of the star-triangle equations. Direct calculation shows that the solution obtained in this way is also valid in the L-even case.

Here we shall use slightly different conventions from [6]. We replace the spectral parameter u and the elliptic nome q^2 by $-\pi u/L$ and p, respectively, and employ the following definition of the Jacobian theta functions (cf (2.6), (2.9) and (2.10) in [6]):

$$\theta_1(u, p) = 2|p|^{1/8} \sin u \prod_{n=1}^{\infty} (1-2p^n \cos 2u + p^{2n})(1-p^n) \tag{2.5a}$$

$$\theta_4(u, p) = \prod_{n=1}^{\infty} (1-2p^{n-1/2} \cos 2u + p^{2n-1})(1-p^n). \tag{2.5b}$$

In the rest of this paper we deal exclusively with a regime specified by $0<p<1$, $-1<u<0$ (regime III in the notation of [1, 5]).

3. Local state probabilities

3.1. *Expressions for the* LSP

Let (b, c) be an admissible pair of the states and consider the alternatively ordered configuration thereof. We shall call such a configuration the ground state of type (b, c) (or equivalently of type (c, b)) (see figure 3).

b	c	b	c
c	b	c	b
b	c	b	c
c	b	c	b

Figure 3. Ground state of type (b, c) on a two-dimensional square lattice.

We consider the probability $P(a\,|\,b, c)$ that a state variable λ_1 takes a given state a under the condition that those far from site 1 are frozen to the ground state of type (b, c). We now introduce parameters ε and x through the relations:

$$\begin{aligned} p &= \exp(-\varepsilon/L) && (\varepsilon>0) \\ x &= \exp(-4\pi^2/\varepsilon) && (0<x<1). \end{aligned} \tag{3.1}$$

By the corner transfer matrix method the LSP $P(a\,|\,b, c)$ is reduced to the $m\to\infty$ limit of the quantity $P_m(a\,|\,b, c)$ given below:

$$P_m(a\,|\,b, c) = \varepsilon_a^L E(-x^a, x^L) X_m(a, b, c; x^2)/S_m(b, c) \tag{3.2a}$$

$$S_m(b, c) = \sum_a \varepsilon_a^L E(-x^a, x^L) X_m(a, b, c; x^2) \tag{3.2b}$$

$$X_m(a, b, c; q) = \sum_\lambda q^{\sum_{j=1}^m jH(\lambda_j, \lambda_{j+1}, \lambda_{j+2})}. \tag{3.2c}$$

Here the a sum in (3.2b) extends over all the states (2.1) with the assumption $x^{\bar\lambda} = x^\lambda$. The outer sum Σ_λ in (3.2c) is taken over the state variables $\lambda_2, \lambda_3, \ldots, \lambda_m$ ($\lambda_1 = a$, $\lambda_{m+1} = b$, $\lambda_{m+2} = c$) under the restriction that $(\lambda_j, \lambda_{j+1})$ is admissible for $1\leq j\leq m$. The symbol ε_λ^l and the quantity $E(z, q)$ are defined as follows:

$$\varepsilon_\lambda^l = \begin{cases} \frac{1}{2} & \text{if } \lambda \text{ or } \bar\lambda \equiv 0 \bmod l \\ 1 & \text{otherwise} \end{cases} \tag{3.3}$$

$$E(z, q) = \prod_{n=1}^\infty (1 - zq^{n-1})(1 - z^{-1}q^n)(1 - q^n). \tag{3.4}$$

For the three states a, b, c, such that (a, b) and (b, c) are admissible, the function $H(a, b, c)$ takes the following form:

$$H(a, b, c) = \tfrac{1}{4}|a - c| \qquad \text{if } a, b, c \neq \bar 0, \bar L \tag{3.5a}$$

$$H(0, 1, \bar 0) = 1 \tag{3.5b}$$

$$H(a, b, c) = H(c, b, a) = H(\bar a, \bar b, \bar c) = H(a^*, b^*, c^*). \tag{3.5c}$$

Note that the $X_m(a, b, c)$, and hence $P_m(a\,|\,b, c)$, vanishes unless $a - b \equiv m \bmod 2$. (The difference $\bar L - 1$ should be interpreted as $L - 1$, etc.)

3.2. Evaluation of $X_m(a, b, c; q)$ in the limit of m large

The quantity $X_m(a, b, c; q)$ introduced in (3.2) is of primary importance in the study of the LSP. Following [5] we call it the one-dimensional configuration sum. By definition it enjoys the following symmetries:

$$X_m(a, b, c) = X_m(a^*, b^*, c^*) = X_m(\bar a, \bar b, \bar c) \tag{3.6a}$$

$$X_m(a, b, c) = X_m(\bar a, b, c) \qquad \text{if } b, c \neq a, \bar a \tag{3.6b}$$

where we have suppressed the argument q. The essential point is that as m goes to infinity the $X_m(a, b, c)$ tends to the modular function appearing in the theta-function identity which we described below.

For $j, l \in \frac{1}{2}Z$, $\varepsilon_1, \varepsilon_2 = \pm 1$, define an elliptic theta function by

$$\theta_{j,l}^{(\varepsilon_1, \varepsilon_2)}(z, q) = \sum_{\substack{n=-\infty \\ \gamma = n + j/2l}}^{\infty} \varepsilon_2^n q^{l\gamma^2}(z^{-l\gamma} + \varepsilon_1 z^{l\gamma}). \tag{3.7}$$

It has the following quasiperiodicities:

$$\theta_{j,l}^{(\varepsilon_1,\varepsilon_2)}(z,q)=\varepsilon_1\theta_{j,l}^{(\varepsilon_1,\varepsilon_2)}(z^{-1},q) \tag{3.8a}$$

$$=\varepsilon_2(zq)^l\theta_{j,l}^{(\varepsilon_1,\varepsilon_2)}(zq^2,q). \tag{3.8b}$$

Let j_1 and j_2 be integers satisfying $0\leqslant j_1\leqslant L-1$, $0<j_2<3$. Theta functions $\{\theta_{j_3,L}^{(+,+)}(z,q)|j_3\in Z, 0\leqslant j_3\leqslant L\}$ form a base of the space that consists of functions having the quasiperiodicities (3.8) with $\varepsilon_1=\varepsilon_2=+1$, $l=L$. This asserts that there exists an identity of the form

$$\theta_{j_1,L-1}^{(+,+)}(z,q)\theta_{j_2,3}^{(-,+)}(z,q)/\theta_{1,2}^{(-,+)}(z,q)=\sum_{j_3}c_{j_1j_2j_3}(q)\theta_{j_3,L}^{(+,+)}(z,q). \tag{3.9}$$

Here the sum is taken over $j_3\in Z$ such that $0\leqslant j_3\leqslant L$, $j_1+j_2+1\equiv j_3 \bmod 2$. The latter condition comes from the fact that $\theta_{j,l}^{(\pm,+)}(z\exp(2\pi \mathrm{i}),q)=(-1)^j\theta_{j,l}^{(\pm,+)}(z,q)$ for $l\in Z$. The identity (3.9) in turn uniquely characterises the entry $c_{j_1j_2j_3}(q)$. In particular, the following automorphic property is valid as a direct consequence of that for the theta functions:

$$c_{j_1j_2j_3}(q)=\sum_{k_1=0}^{L-1}\sum_{k_2=1}^{2}\sum_{k_3=0}^{L}\left(\frac{8}{3L(L-1)}\right)^{1/2}\varepsilon_{k_1}^{L-1}\varepsilon_{j_3}^{L}$$
$$\times\cos\left(\frac{\pi k_1 j_1}{L-1}\right)\sin\left(\frac{\pi k_2 j_2}{3}\right)\cos\left(\frac{\pi k_3 j_3}{L}\right)c_{k_1k_2k_3}(t) \tag{3.10}$$

where the conjugate modulus t is related to q through the relation

$$q=\exp(2\pi \mathrm{i}\tau)\qquad t=\exp(-2\pi \mathrm{i}/\tau). \tag{3.11}$$

The modular invariance (3.10) and (3.11) plays a key role in § 4 in the study of critical behaviours for the LSP.

Now we give the formulae by which the one-dimensional configuration sums are identified with the modular functions $c_{j_1j_2j_3}(q)$. As is done in [5], these are verified by rewriting the multiple sum expression (3.2) in a series involving Gaussian polynomials [8] and taking the straightforward limit $m\to\infty$. In the following we fix the parity of m to be even. The odd-m limit can be reduced to this case:

$$\lim_{m\ \text{even}\to\infty}\tfrac{1}{2}(X_m(a,b,c;q)+X_m(\bar{a},b,c;q))$$
$$=\varepsilon_r^{L-1}q^{\nu(r,s,a)}c_{r,s,a}(q)\qquad \text{if } a,b,c\neq\bar{0},\bar{L}. \tag{3.12a}$$

$$\lim_{m\ \text{even}\to\infty}(X_m(0,0,1;q)-X_m(\bar{0},0,1;q))=\phi(q)/\phi(q^2) \tag{3.12b}$$

where the variables r and s are determined from b and c by

$$r=\frac{b+c-1}{2}\qquad s=\frac{b-c+1}{2}+1. \tag{3.13}$$

The functions $\nu(j_1,j_2,j_3)$ and $\phi(q)$ are defined as follows:

$$\nu(j_1,j_2,j_3)=-\frac{1}{8}+\frac{j_1+j_2-j_3}{4}-\frac{j_1^2}{4(L-1)}-\frac{j_2^2}{12}+\frac{j_3^2}{4L} \tag{3.14}$$

$$\phi(q)=\prod_{n=1}^{\infty}(1-q^n). \tag{3.15}$$

The formulae (3.12)-(3.15) along with the symmetries (3.6) yield the evaluation of the one-dimensional configuration sums as modular functions.

3.3. *The results for the* LSP

Much the same as the one-dimensional configuration sum $X_m(a, b, c)$, LSP itself enjoys the symmetries

$$P(a \mid b, c) = P(a^* \mid b^*, c^*) = P(\bar{a} \mid \bar{b}, \bar{c}) \tag{3.16a}$$

$$P(a \mid b, c) = P(\bar{a} \mid b, c) \qquad \text{if } b, c \neq a, \bar{a}. \tag{3.16b}$$

Below we present the result for the LSP. Thanks to the symmetries (3.16), LSP for general (a, b, c) are reduced to these cases:

$$\tfrac{1}{2}(P(a \mid b, c) + P(\bar{a} \mid b, c))$$

$$= \varepsilon_a^L \frac{\theta_{1,2}^{(-,+)}(x, x^2)\,\theta_{a,L}^{(+,+)}(x, x^2)}{\theta_{s,3}^{(-,+)}(x, x^2)\,\theta_{r,L-1}^{(+,+)}(x, x^2)} c_{r,s,a}(x^2) \qquad \text{if } a, b, c \neq \bar{0}, \bar{L} \tag{3.17a}$$

$$P(0 \mid 0, 1) - P(\bar{0} \mid 0, 1) = \frac{E(-1, x^L)\phi(x)}{E(-1, x^{L-1})\phi(x^4)}. \tag{3.17b}$$

The symbol ε_a^L and the variables r, s have been respectively defined in (3.3) and (3.13). These results are obtained by specialising the identity (3.9) to $z = x$, $q = x^2$ and dividing by the LHS. They are identified with the $m \to \infty$ limit of the expression (3.2) by using the formulae (3.12)-(3.15) and an identity

$$x^{l/8}\theta_{j,l}^{(\varepsilon_1,\varepsilon_2)}(x, x^2) = x^{(l-2j)^2/8l} E(-\varepsilon_1 x^j, \varepsilon_2 x^l). \tag{3.18}$$

Clearly we see the condition is satisfied that the total probability should be unity.

4. Critical behaviour

Our model becomes critical as the parameter p tends to zero or, equivalently, x to unity (see (3.1)). By the inversion method [7] it is straightforward to compute the free energy. The result coincides with that for regime III of the 8VSOS models [1] with r being L. An explicit expression is contained in [5] as the case $N = 1$. From this we observe that the specific heat critical exponent α has the value

$$2 - \alpha = \tfrac{1}{2}L. \tag{4.1}$$

In the following we study the critical behaviour of the LSP obtained in the previous section. For the purpose we first rewrite the expression (3.17) in a form that is suitable for examining the small-p behaviour. This is achieved by the modular transformation (3.10) and (3.11) that interchanges the elliptic nomes q and t. From the relations (3.1), (3.11) and $q = x^2$ as in (3.17) we readily see

$$t = p^{L/2}. \tag{4.2}$$

Thus the parameter t serves as an appropriate 'deviation from criticality' variable. We can immediately read off the exponents

$$\Delta = \beta/(2 - \alpha) \tag{4.3}$$

by counting the powers occurring in the small-t expansion of the LSP. Note that by the scaling hypothesis Δ is related to the anomalous dimension η through $\eta = 4\Delta$.

Besides the formula (3.10) we apply so-called conjugate modulus identities for the theta functions in (3.17). These are given as follows:

$$x^{l/8}\theta_{j,l}^{(-,+)}(x, x^2)=\left(\frac{\varepsilon}{2\pi l}\right)^{1/2}\theta_1(\pi j/l, t^{2/l}) \tag{4.4a}$$

$$x^{l/8}\theta_{j,l}^{(+,+)}(x, x^2)=\left(\frac{\varepsilon}{2\pi l}\right)^{1/2}t^{1/8l}\frac{\theta_1(\pi j/l, t^{1/l})}{\theta_1(\pi j/l, t^{2/l})}\frac{\phi(t^{2/l})^2}{\phi(t^{1/l})} \tag{4.4b}$$

$$x^{l/8}E(-1, x^l)=\left(\frac{\varepsilon}{2\pi l}\right)^{1/2}t^{1/8l}\phi(t^{1/l})^2/\phi(t^{2/l}) \tag{4.4c}$$

$$x^{l/24}\phi(x^l)=\left(\frac{\varepsilon}{2\pi l}\right)^{1/2}t^{1/12l}\phi(t^{2/l}). \tag{4.4d}$$

Utilising (3.10) and (4.4) we rewrite the LSP (3.17) in suitable forms:

$$\begin{aligned}
&\tfrac{1}{2}(P(a\,|\,b, c)+P(\bar{a}\,|\,b, c))\\
&\quad=\frac{2}{L}(\varepsilon_a^L)^2\frac{\phi(t^{2/L})^2\phi(t^{1/(L-1)})}{\phi(t^{1/L})\phi(t^{2/(L-1)})^2}\\
&\quad\times t^{-1/8L(L-1)}\frac{\theta_1(\pi/2, t)\theta_1(\pi a/L, t^{1/L})\theta_1(\pi a/(L-1), t^{2/(L-1)})}{\theta_1(\pi s/3, t^{2/3})\theta_1(\pi a/L, t^{2/L})\theta_1(\pi r/(L-1), t^{1/(L-1)})}\\
&\quad\times\sum_{k_1=0}^{L-1}\sum_{k_2=1}^{2}\sum_{k_3=0}^{L}\varepsilon_{k_1}^{L-1}\cos\left(\frac{\pi r k_1}{L-1}\right)\\
&\quad\times\sin\left(\frac{\pi s k_2}{3}\right)\cos\left(\frac{\pi a k_3}{L}\right)c_{k_1k_2k_3}(t) \qquad \text{if } a, b, c\neq\bar{0}, \bar{L}
\end{aligned} \tag{4.5}$$

$$P(0|0, 1)-P(\bar{0}|, 1)=2\left(\frac{L-1}{L}\right)^{1/2}t^{\Delta_{P_0-P_{\bar{0}}}}\frac{\phi(t^2)\phi(t^{1/L})^2\phi(t^{2/(L-1)})}{\phi(t^{1/2})\phi(t^{1/(L-1)})^2\phi(t^{2/L})} \tag{4.6a}$$

where the exponent $\Delta_{P_0-P_{\bar{0}}}$ is

$$\Delta_{P_0-P_{\bar{0}}}=\frac{(L+1)(L-2)}{16L(L-1)}. \tag{4.6b}$$

From (4.5) and (4.6) we find that, as t tends to zero, the LSP $P(a|b, c)$ becomes independent of (b, c) and converges to the critical value

$$P_a^{(c)}=2(\varepsilon_a^L)^2/L. \tag{4.7}$$

Now we can readily evaluate the leading powers of t appearing in the LSP. By virtue of (4.5) and (2.5a) they are given by

$$\tfrac{1}{24}+\text{the lowest power of } c_{k_1k_2k_3}(t). \tag{4.8}$$

Taking advantage of the relations (3.12)-(3.14) and the fact that

$$\lim_{m\text{ even}\to\infty}X_m(a, b, c; t)=t^{(a-b)(a-c)/4}(1+\mathrm{O}(t)) \qquad \text{if } a, b, c\neq\bar{0}, \bar{L} \tag{4.9}$$

we find the lowest power of $c_{k_1k_2k_3}(t)$ $(0\leqslant k_1\leqslant L-1, 0<k_2<3, 0\leqslant k_3\leqslant L, k_1+k_2\equiv k_3+1, \bmod 2)$ in the following form:

$$-\tfrac{1}{24}+\Delta_{k_1,k_3} \tag{4.10a}$$

Table 1. The exponents $\Delta_{p,q}$ in (4.10b). The subscript p (respectively q) runs horizontally (respectively vertically) from the bottom left ($p=q=0$).

(a) $L=3$.

$\frac{3}{2}$	$\frac{3}{8}$	0
$\frac{2}{3}$	$\frac{1}{24}$	$\frac{1}{6}$
$\frac{1}{6}$	$\frac{1}{24}$	$\frac{2}{3}$
0	$\frac{3}{8}$	$\frac{3}{2}$

(b) $L=4$.

3	$\frac{4}{3}$	$\frac{1}{3}$	0
$\frac{27}{16}$	$\frac{25}{48}$	$\frac{1}{48}$	$\frac{3}{16}$
$\frac{3}{4}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{3}{4}$
$\frac{3}{16}$	$\frac{1}{48}$	$\frac{25}{48}$	$\frac{27}{16}$
0	$\frac{1}{3}$	$\frac{4}{3}$	3

(c) $L=5$.

5	$\frac{45}{16}$	$\frac{5}{4}$	$\frac{5}{16}$	0
$\frac{16}{5}$	$\frac{121}{80}$	$\frac{9}{20}$	$\frac{1}{80}$	$\frac{1}{5}$
$\frac{9}{5}$	$\frac{49}{80}$	$\frac{1}{20}$	$\frac{9}{80}$	$\frac{4}{5}$
$\frac{4}{5}$	$\frac{9}{80}$	$\frac{1}{20}$	$\frac{49}{80}$	$\frac{9}{5}$
$\frac{1}{5}$	$\frac{1}{80}$	$\frac{9}{20}$	$\frac{121}{80}$	$\frac{16}{5}$
0	$\frac{5}{16}$	$\frac{5}{4}$	$\frac{45}{16}$	5

where the quantity Δ_{k_1,k_3} is defined by

$$\Delta_{k_1,k_3}=\Delta_{L-1-k_1,L-k_3}=\frac{[Lk_1-(L-1)k_3]^2}{4L(L-1)} \qquad 0\leqslant k_1\leqslant L-1 \qquad 0\leqslant k_3\leqslant L. \tag{4.10b}$$

From (4.8) and (4.10) we conclude that the leading corrections to the LSP as t tends to zero consist of the terms proportional to $t^{\Delta_{k_1,k_3}}$. The exponents $\Delta_{p,q}$ for $L=3$, 4 and 5 are listed in table 1.

5. Summary and discussion

In this paper we have exactly computed the local state probabilities (LSP) for a series of solvable interaction-round-face models. The LSP is neatly expressed in (3.17) by exploiting the theta function identity (3.9). Critical exponents α and $\Delta=\beta/(2-\alpha)$ are determined in (4.1) and (4.6b), (4.10b), respectively.

The characteristic feature of our model emerges in the relevant identity (3.9). It contrasts with the one appearing in regime III of the 8VSOS model viewed as the $N=1$ case of more general models in [5]. The latter has the form (3.9) with the $\theta^{(+;+)}_{j_1,L-1}(z,q)$ (respectively $\theta^{(+;+)}_{j_3,L}(z,q)$) replaced by $\theta^{(-;+)}_{j_1,L-1}(z,q)$ (respectively $\theta^{(-;+)}_{j_3,L}(z,q)$). This results in the difference of the exponents (4.10b) from those for the 8VSOS model where the numerator takes the form $[Lk_1-(L-1)k_3]^2-1$ and k_1 and k_3 are restricted to $0<k_1<L-1$, $0<k_3<L$.

As in [5] we can consider the lowest power of the modular function $c_{k_1k_2k_3}(t)$ as $-c/24$, where the c is the central charge of the conformal field theory to which the model renormalises. From (4.10) we find that our model has $c=1$ for all $L\geqslant 3$. This is consistent with the fact that the $L=3$ case coincides with the odd-height sector of the SOS models in [5] with $(L, N)=(6, 2)$, which is known to correspond to $N=1$ superconformal field theory with $c=1$.

Apart from the relevant theta-function identities, our model and the 8VSOS model may be compared by using the state variables and the conditions on their adjacent pairs. Just as the former is characterised by figure 1, so is the latter by the diagram in figure 4.

Figure 4. Diagram for the restricted eight-vertex SOS model. Each vertex (open circle) corresponds to a local state (called 'height' in [1]). Two states can occupy nearest-neighbour lattice sites if they are connected on the diagram.

Figure 4 is the Dynkin diagram for the classical Lie algebra A_{L-1} as pointed out by Pasquier [9]. He proposed the models labelled by simply laced classical Lie algebras and a method to get trigonometric solutions to their star-triangle equations. The approach is based on the Temperley-Lieb algebraic structure of local transfer matrices (as for the 8VSOS model, see also [10]). Our model originally introduced as a 'special S_2 generalisation' [11] of the 8VSOS model corresponds to the affine Lie algebra $D^{(1)}_{L+2}$ in this picture. We remark that the critical LSP $P^{(c)}_a$ in (4.7) coincides with a square of a component of properly normalised Perron-Frobenius vector for $D^{(1)}_{L+2}$ in agreement with Pasquier's argument. We hope to discuss this point in a future publication.

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